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The Whitham deformation of the Dijkgraaf-Vafa theory

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Abstract: We discuss the Whitham deformation of the effective superpotential in the Dijkgraaf-Vafa (DV) theory. It amounts to discussing the Whitham deformation of an underlying (hyper)elliptic curve. Taking the elliptic case for simplicity we derive the Whitham equation for the period, which governs flowings of branch points on the Riemann surface. By studying the hodograph solution to the Whitham equation it is shown that the effective superpotential in the DV theory is realized by many different meromorphic differentials. Depending on which meromorphic differential to take, the effective superpotential undergoes different deformations. This aspect of the DV theory is discussed in detail by taking the $N = 1^*$ theory. We give a physical interpretation of the deformation parameters.

Keywords: Topological Field Theories, Nonperturbative Effects, Integrable Hierarchies, Matrix Models.

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1. Introduction

One of the long-standing problems in the quantum field theory is to understand non-perturbative dynamics of the supersymmetric QCD. The recent works by Dijkgraaf and Vafa [1, 2] have given a breakthrough towards to this direction. They found a close relation between the $N = 1$ supersymmetric QCD and a matrix model. It is now confirmed by subsequent works by many people. This Dijkgraaf-Vafa theory revived a renewed attention [3, 4] to the Seiberg-Witten theory (SW) for the $N = 2$ supersymmetric QCD [5]. Indeed both theories have relevance to the Riemann surface and the Whitham hierarchy as commonly underlying features. To be concrete, take the case which is concerned about the Riemann surface with genus one. The DV theory gives an effective superpotential of the $N = 1$ supersymmetric QCD

$$W_{DV} = \int d^2\theta \left(N \frac{\partial \mathcal{F}_0}{\partial S} - 2\pi i \overset{\circ}{\tau} S \right), \tag{1.1}$$

which is the perturbative part of the Veneziano-Yankielovicz superpotential. On the other hand the SW theory gives the effective action of the $N = 2$ supersymmetric QCD

$$W_{SW} = \frac{1}{4\pi} \mathrm{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}_0(A)}{\partial A} \bar{A} + \int d^2\theta \, \frac{1}{2} \tau_0(A) W_\alpha W^\alpha \right]. \tag{1.2}$$

In (1.1) and (1.2) $\mathcal{F}_0$ is called the free energy and characterized by a certain differential on an elliptic curve. In (1.2) $\tau_0(A)$ is the period of the curve.

Since the SW theory was born, the Whitham hierarchy attracted attention as a underlying integrable structure in the SW theory [6, 7, 8]. But dynamical aspects of the hierarchy was not properly studied. Dynamical flowings of branch points of the Riemann surface and the consequent creation or annihilation of branch cuts were the original concerns to study the Whitham hierarchy [9]. There people aimed regulation (or modulation) of singular behaviors of finite-gap solutions for the KdV system. We think it important to shed more light on this aspect of the DV and SW theories in the revived era of the interest. In this paper we study the Whitham deformation of the elliptic curve as a flow of the branch points on the Riemann surface. We will show that the flow is governed by the Whitham equation for the period τ of the curve (4.5), i.e.,

$$\frac{\partial \tau(a, \vec{T})}{\partial T_M} = s_M(\tau(a, \vec{T})) \frac{\partial \tau(a, \vec{T})}{\partial a}, \qquad \tau(a, \vec{0}) \equiv \tau_0(a). \tag{1.3}$$

Here $\vec{T} = (T_1, T_2, \ldots)$ and a is either S for the DV theory or A for the SW theory. $s_1(\tau(a, \vec{T})), s_2(\tau(a, \vec{T})), \ldots$, are the characteristic speeds. (For more explanations see the discussion thereabout.) It is noteworthy that the Whitham equation looks like the renormalization group equation for the running coupling

$$\frac{\partial \bar{g}(t, g)}{\partial t} = \beta(g) \frac{\partial \bar{g}(t, g)}{\partial g}.$$

In the recent papers [10] they developed interesting arguments on an underlying integrable structure of the DV theory. Namely they discussed that the equilibria of the superpotential of the DV theory, (1.1), correspond to the stable flow points of some integrable systems at which the relevant (hyper)elliptic curve degenerates. But the flow is not the one given by the Whitham deformation (1.3), which is of interest in this paper.

It is known [11] that the equation of the type (1.3) admits a hodograph solution as a special solution and it is characterized by a free energy $\mathcal{F}$. We will find that the initial condition $\tau_0(a)$ of a hodograph solution is imposed by specifying the inverse function $a(\tau_0) (\equiv \tau_0^{-1}(a))$ as (5.6), i.e.,

$$a(\tau(a, \vec{0})) = \sum_{M=1}^{M_0} \Lambda_M s_M(\tau(a, \vec{0})). \tag{1.4}$$

It will be shown that (1.4) is given as a period integral of a certain differential, which is nothing but the DV or SW differential. Then the free energy $\mathcal{F}_0$ in (1.1) and (1.2) is an initial value of the free energy $\mathcal{F}$ which characterizes the hodograph solution

$$\mathcal{F}_0(a) = \mathcal{F}|_{\vec{T}=0}\,. \tag{1.5}$$

Choosing a set of non-vanishing Λ_M in (1.4) uniquely specifies the initial condition $\tau_0(a)$ and therefore the free energy $\mathcal{F}_0(a)$. Here the characteristic speeds $s_M(\tau(a,\vec{0}))$, $M = 1, 2, \ldots$, were kept fixed. Now we reverse the argument. Namely we fix the initial condition $\tau_0(a)$, that is, the l.h.s. of (1.4), and specify the characteristic speeds so as to satisfy (1.4) for the chosen set of $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{M_0})$. We show that it can be done by changing parameterization of the elliptic curve. Then the fixed initial condition $\tau_0(a)$ is given by different DV or SW differentials. The hodograph solution $\tau(a,\vec{T})$ is characterized by different free energies $\mathcal{F}$. But the key point is that its initial value $\mathcal{F}_0(a)$ remains fixed as $\tau_0(a) = \frac{\partial^2 \mathcal{F}_0}{\partial a^2}$. In other words a fixed $\mathcal{F}_0(a)$ undergoes different Whitham deformations depending on the choice of $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{M_0})$. To discuss concretely this aspect of the DV theory, we will take the $N = 1^*$ theory [2, 12] as an example. We will explicitly give the superpotential of the theory by various DV differentials, and show the possibility of different Whitham deformations.

The paper is organized as follows. In section 2 we define quantities τ_{MN} as extensions of the period $\tau(\equiv \tau_{00})$ of the elliptic curve. They are important constituents in the paper. In section 3 we discuss on a "gauge" freedom in parameterizing the elliptic curve. Fixing this freedom we find an equation for the curve, which plays a fundamental role in the paper. In section 4 we think of deforming the curve. We introduce the Whitham hierarchy to the deformation. It is shown to amount to assuming the Whitham equation (1.3). The Whitham hierarchy is mimic to the dispersionless KP hierarchy in the topological field theory [13, 14, 15]. Therefore it may be formulated by means of analogous quantities with the 2-point functions of the topological field theory. We find them by modifying τ_{MN}, discussed in section 2. In section 5 we study the hodograph solution to the Whitham equation according to [11, 15]. We then interpret the solution in terms of the topological field theory, *i.e.*, by employing the terminology like the free energy, the *small* or *large phase space etc.* In section 6 the dual version of the Whitham equation is discussed. The initial condition for the hodograph solution is characterized by the DV or SW differential. In the basis of these arguments we discuss various Whitham deformations of the $N = 1^*$ theory in section 7. In section 8 we give a matrix-model interpretation of those Whitham deformations. Appendix A is devoted to give a short summary on the elliptic curve and some useful formulae for the period integral. In appendix B we discuss a systematic method to evaluate the 2-point functions τ_{MN}. In appendix C we give some calculations to check the consistency of the formalism for the Whitham hierarchy, developed in this paper.

2. The elliptic curve

Throughout the paper we consider an elliptic curve defined by

$$y^2 = 4(x-u)(x-v)(x-w)\,. \tag{2.1}$$

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It may be also given by

$$y^2 = 4(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)\,.$$

But this reduces to (2.1) by the change

$$\frac{1}{x-\lambda_4} \longrightarrow x\,, \qquad \frac{y}{\sqrt{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}} \longrightarrow \frac{y}{x^2}\,.$$

On the curve (2.1) there exists one holomorphic differential $d\omega_0$ of the form

$$d\omega_0 = \frac{1}{g_0}\frac{dx}{y}\,. \tag{2.2}$$

By the normalization

$$\oint_A d\omega_0 = 1\,, \tag{2.3}$$

g_0 is fixed to be

$$g_0 = \oint_A \frac{dx}{y}\,. \tag{2.4}$$

The period on the curve is given by

$$\tau = \oint_B d\omega_0 = \frac{g_{D0}}{g_0}\,, \tag{2.5}$$

with

$$g_{D0} = \oint_B \frac{dx}{y}\,.$$

On the curve there exists also a set of meromorphic differentials $d\Omega_M$, $M = 1, 2, \ldots$, of the form

$$\begin{aligned} d\Omega_M &= [x^M + \gamma_{M-1}x^{M-1} + \cdots + \gamma_1 x + \gamma_0]\frac{dx}{y} \\ &\equiv Q_M(x)\frac{dx}{y}\,. \end{aligned} \tag{2.6}$$

The coefficients $\gamma_{M-1}, \ldots, \gamma_0$ of the polynomial Q_M is uniquely determined by requiring $d\Omega_M$ to satisfy

$$\oint_A d\Omega_M = 0\,, \tag{2.7}$$

and to have poles at $x = \frac{1}{\xi^2} = \infty$ such that

$$d\Omega_M = -\xi^{-2M}d\xi + \text{holomorphic}\,. \tag{2.8}$$

It is done by the recursive formula (B.3) and (B.6).

Let us introduce the following quantities as an extension of the period (2.5):

$$\begin{aligned} \tau_{0M} &\equiv \tau_{M0} \\ &= \oint_B d\Omega_M\,, \qquad M = 1, 2, \ldots\,. \end{aligned} \tag{2.9}$$

It becomes

$$\tau_{0M} = -2\pi i \mathrm{res}_{\xi=0}\left[\frac{\xi^{-2M+1}}{2M-1} d\omega_0\right], \tag{2.10}$$

by the Riemann bilinear relation. By the analogy we also introduce the quantities

$$\tau_{NM} = -2\pi i \ \mathrm{res}_{\xi=0}\left[\frac{\xi^{-2M+1}}{2M-1} d\Omega_N\right], \qquad M, N = 1, 2, \dots . \tag{2.11}$$

Using the Riemann bilinear relation again we have

$$\tau_{NM} = \tau_{MN} \,. \tag{2.12}$$

These quantities will be important constituents in our discussions later.

3. Gauge fixing and the fundamental equation

Let us assume that the branch points of (2.1) are parameterized by one parameter alone, namely the period as $u(\tau), v(\tau), w(\tau)$. In general an elliptic curve is determined by g_0 and g_{D0}. The assumption implies that they are functions of τ. To realize the assumption it suffices to set g_0 to be a particular function of τ, because they are related by (2.5). We call it "gauge fixing", which is an abusive terminology. For instance, with the "gauge" $g_0 = 2\pi$ the branch points of the curve are expressed as

$$\begin{aligned} u(\tau) &= \frac{c}{3} + \frac{1}{12}[\theta_3(\tau)^4 + \theta_0(\tau)^4] \,, \\ v(\tau) &= \frac{c}{3} + \frac{1}{12}[\theta_2(\tau)^4 - \theta_0(\tau)^4] \,, \\ w(\tau) &= \frac{c}{3} - \frac{1}{12}[\theta_2(\tau)^4 + \theta_3(\tau)^4] \,, \end{aligned} \tag{3.1}$$

by using the Weierstrass standard form (A.1) and (A.2). Here c is given by

$$c = u(\tau) + v(\tau) + w(\tau) \,, \tag{3.2}$$

which is still to be fixed arbitrarily. We can also take the "gauge" which sets one of the branch points, for instance, $w(\tau) = 1$. Then we have

$$\begin{aligned} u(\tau) &= \frac{c}{3} + \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 [\theta_3(\tau)^4 + \theta_0(\tau)^4] \,, \\ v(\tau) &= \frac{c}{3} + \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 [\theta_2(\tau)^4 - \theta_0(\tau)^4] \,, \end{aligned} \tag{3.3}$$

where g_0 is given by

$$1 = \frac{c}{3} - \frac{1}{12}\left(\frac{\pi}{g_0}\right)^2 [\theta_2(\tau)^4 + \theta_3(\tau)^4] \,.$$

With the former "gauge" the three branch points $u(\tau), v(\tau), w(\tau)$ move depending τ. Instead with the latter one only two of them move.

When the branch points $u(\tau), v(\tau), w(\tau)$ are given as such, we can show the fundamental equation throughout the paper:

$$\frac{\partial}{\partial\tau} d\Omega_M(x) = s_M(\tau)\frac{\partial}{\partial\tau} d\omega_0 + d[\Delta_M(x,\tau)]\,, \qquad M = 1, 2, \ldots\,. \tag{3.4}$$

Here $s_M(\tau)$ in the first term is the characteristic speed given by

$$s_M(\tau) = g_0 \frac{Q_M(u)u'(v-w) + Q_M(v)v'(w-u) + Q(w)_M w'(u-v)}{u'(v-w) + v'(w-u) + w'(u-v)}\,, \tag{3.5}$$

and the second term is an exact form with

$$\begin{aligned} \Delta_M(x,\tau) = &- \frac{Q_M(u)u'(v'w - w'v) + Q_M(v)v'(w'u - u'w) + Q_M(w)w'(u'v - v'u)}{u'(v-w) + v'(w-u) + w'(u-v)}\frac{1}{y} + \\ &+ \frac{Q_M(u)u'(v' - w') + Q_M(v)v'(w' - u') + Q_M(w)w'(u' - v')}{u'(v-w) + v'(w-u) + w'(u-v)}\frac{x}{y}\,. \end{aligned} \tag{3.6}$$

To prove (3.4), first of all note that (2.8) implies that $\frac{\partial}{\partial\tau} d\Omega_M(x)$ is holomorphic. Therefore we have

$$\begin{aligned} \frac{\partial}{\partial\tau} d\Omega_M(x) &= \frac{\partial Q_M(x)}{\partial\tau}\frac{dx}{y} + Q_M(x)\frac{\partial}{\partial\tau}\frac{dx}{y} \\ &= \left[A_M(u,v,w) + \frac{1}{2}\left(\frac{u'Q_M(u)}{x-u} + \frac{v'Q_M(v)}{x-v} + \frac{w'Q_M(w)}{x-w}\right)\right]\frac{dx}{y}\,, \end{aligned} \tag{3.7}$$

with some functions $A_M(u,v,w)$. On the other hand we also note that

$$\begin{aligned} \frac{1}{x-v}\frac{dx}{y} &= \frac{1}{w-v}\left(1 - \frac{w-u}{x-u}\right)\frac{dx}{y} + d\left[\frac{2}{v-w}\left(\frac{w}{y} - \frac{x}{y}\right)\right], \\ \frac{1}{x-w}\frac{dx}{y} &= \frac{1}{v-w}\left(1 - \frac{v-u}{x-u}\right)\frac{dx}{y} + d\left[\frac{2}{w-v}\left(\frac{v}{y} - \frac{x}{y}\right)\right], \end{aligned} \tag{3.8}$$

by calculating $d(1/y)$ and $d(x/y)$. Plugging these into the equation, obtained by calculating $\frac{\partial}{\partial\tau}\frac{dx}{y}$ similarly to (3.7), we find

$$\frac{\partial}{\partial\tau}\frac{dx}{y} = \alpha(u,v,w)\frac{dx}{y} + \beta(u,v,w)\frac{1}{x-u}\frac{dx}{y} + d[\delta(u,v,w)]\,, \tag{3.9}$$

with

$$\begin{aligned} \alpha(u,v,w) &= -\frac{1}{2}\frac{v'-w'}{v-w}\,, \\ \beta(u,v,w) &= \frac{1}{2}\left(u' - v'\frac{w-u}{w-v} - w'\frac{v-u}{v-w}\right), \\ \delta(u,v,w) &= \frac{v'w - w'v}{v-w}\frac{1}{y} - \frac{v'-w'}{v-w}\frac{x}{y}\,. \end{aligned}$$

By (3.9) as well as those obtained by replacing u, v and w cyclically, (3.7) becomes

$$\frac{\partial}{\partial\tau} d\Omega_M(x) = \tilde{A}_M(u,v,w)\frac{dx}{y} + B_M(u,v,w)\frac{\partial}{\partial\tau}\frac{dx}{y} + d[C_M(u,v,w)]\,, \tag{3.10}$$

with

$$\begin{aligned}
\tilde{A}_M(u,v,w) &= A_M(u,v,w) - \frac{1}{2}\left[u' Q_M(u)\frac{\alpha(u,v,w)}{\beta(u,v,w)} + \text{cyclic}\right], \\
B_M(u,v,w) &= \frac{1}{2}\left[u' Q_M(u)\frac{1}{\beta(u,v,w)} + \text{cyclic}\right], \\
C_M(u,v,w) &= -\frac{1}{2}\left[u' Q_M(u)\frac{\delta(u,v,w)}{\beta(u,v,w)} + \text{cyclic}\right].
\end{aligned}$$

Integrating (3.10) along a A-cycle yields

$$\tilde{A}_M(u,v,w) = -B_M(u,v,w)\frac{g_0'}{g_0}, \qquad M = 1,2,\dots,$$

owing to (2.7). Plug this into (3.10). Then we obtain the fundamental equation (3.4).

4. The Whitham deformation

We consider a deformation of a "gauge"-fixed curve

$$y^2 = 4(x - u(\tau))(x - v(\tau))(x - w(\tau)), \tag{4.1}$$

through that of τ:

$$\tau \longrightarrow \tau(a, T_1, T_2, \dots) \equiv \tau(a, \vec{T}),$$

with flow parameters a and T_M, $M = 1,2,\dots$. The Whitham deformation is defined by

$$\frac{\partial}{\partial T_M} d\omega_0 = \frac{\partial}{\partial a} d\tilde{\Omega}_M, \tag{4.2}$$

$$\frac{\partial}{\partial T_M} d\tilde{\Omega}_N = \frac{\partial}{\partial T_N} d\tilde{\Omega}_M, \tag{4.3}$$

in which $d\tilde{\Omega}_M$ is a modified meromorphic differential of $d\Omega_M$, (2.6), as

$$d\tilde{\Omega}_M = d\Omega_M - d\left(\int^{\tau} d\tau\, \Delta_M\right), \tag{4.4}$$

with Δ_M given by (3.6). The compatibility of the deformation can be seen by showing that both (4.2) and(4.3) are equivalent to the equation

$$\frac{\partial}{\partial T_M}\tau(a,\vec{T}) = s_M(\tau)\frac{\partial}{\partial a}\tau(a,\vec{T}). \tag{4.5}$$

It is called the Whitham equation. We calculate the r.h.s. of (4.2) with (3.4):

$$\frac{\partial}{\partial a} d\tilde{\Omega}_M = \frac{\partial \tau}{\partial a}\frac{\partial}{\partial \tau} d\tilde{\Omega}_M = \frac{\partial \tau}{\partial a} s_M(\tau)\frac{\partial}{\partial \tau} d\omega_0,$$

which is equal to the l.h.s. of (4.2) due to the Whitham equation (4.5). Similarly we calculate the l.h.s. of (4.3) by (3.4) and (4.5):

$$\frac{\partial}{\partial T_M} d\tilde{\Omega}_N = \frac{\partial \tau}{\partial T_M}\frac{\partial}{\partial \tau} d\tilde{\Omega}_N = s_M(\tau) s_N(\tau)\frac{\partial}{\partial a} d\omega_0.$$

This implies (4.3). Thus all of the flows defined by (4.2) and(4.3) are compatible and they are equivalent to the the single equation (4.5).

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By (2.5), (2.9)–(2.11) and (4.4), we write (4.2) and(4.3) as

$$\frac{\partial\tau}{\partial T_M} = \frac{\partial\tau_{0M}}{\partial a}, \tag{4.6}$$

$$\frac{\partial\tau_{0K}}{\partial T_M} = \frac{\partial\tilde{\tau}_{MK}}{\partial a}, \tag{4.7}$$

$$\frac{\partial\tilde{\tau}_{NK}}{\partial T_M} = \frac{\partial\tilde{\tau}_{MK}}{\partial T_N}, \tag{4.8}$$

with

$$\begin{aligned}\tilde{\tau}_{NM} &= -2\pi i\ \mathrm{res}_{\xi=0}\left[\frac{\xi^{-2M+1}}{2M-1}d\tilde{\Omega}_N\right]\\ &= \tau_{NM} + 2\pi i\ \mathrm{res}_{\xi=0}\left[\xi^{-2M}d\xi\int^{\tau}d\tau\,\Delta_N(x,\tau)\right],\end{aligned} \tag{4.9}$$

for $M, N = 1, 2, \ldots$. By the Riemann bilinear relation we have also for $\tilde{\tau}_{NM}$

$$\tilde{\tau}_{NM} = \tilde{\tau}_{MN}.$$

(4.6)–(4.8) imply that $\tilde{\tau}_{MN}$ together with $\tau(\equiv \tau_{00})$ and $\tau_{0M}(\equiv \tilde{\tau}_{0M})$ are integrable as

$$\tilde{\tau}_{AB} = \frac{\partial^2\mathcal{F}}{\partial T_A \partial T_B}, \qquad A, B = 0, 1, 2, \ldots, \tag{4.10}$$

with a function $\mathcal{F}$ of $T_0(= a)$ and $\vec{T}$. This Whitham hierarchy has exactly the same integrable structure as the dispersionless KP hierarchy for the topological Landau-Ginzburg theory [13, 14]. Employing the terminology in the latter theory we call $\mathcal{F}$ the free energy and $\tilde{\tau}_{AB}$ the 2-point function.

Thus the Whitham hierarchy is mimic to the dispersionless KP hierarchy. In the rest of this section and the next section we proceed the arguments by taking the close analogy with the topological field theory.

5. The hodograph solution

In this section we solve the Whitham equation (4.5) in order to see the flow. It is known in [11, 15] that for an equation of the sort (4.5) we have a hodograph solution such as given by

$$\tau(a, \vec{T}) \equiv \hat{\tau}(\hat{a}), \tag{5.1}$$

where

$$\hat{\tau}(a) \equiv \tau(a, \vec{0}), \tag{5.2}$$

$$\hat{a} = a + \sum_{M=1}^{\infty} T_M s_M(\tau(a, \vec{T})). \tag{5.3}$$

To show this write (4.5) in the form

$$\left[\frac{\partial}{\partial T_M} - s_M(\tau)\frac{\partial}{\partial a}\right]\tau = 0. \tag{5.4}$$

This implies that τ is constant along the characteristic

$$\frac{d\,T_M}{-1} = \frac{da}{s_M(\tau)}\,, \tag{5.5}$$

which is a straight line. Therefore (5.1) with (5.2) and (5.3) is a solution of the Whitham equation. In (5.2) $\tau(a,\vec{0})$ is an arbitrary function. Impose the relation

$$a = \sum_{M=1}^{M_0} \Lambda_M s_M(\tau(a,\vec{0}))\,, \tag{5.6}$$

with a finite number of constants Λ_M, $M = 1, 2, \ldots, M_0$. Then $\tau(a,\vec{0})$ is given by inverting (5.6). The hodograph solution (5.1) is an implicit solution. It still has dependence on the solution $\tau(a,\vec{T})$ itself through $\hat{a}$. An explicit form of the solution is obtained in a formal series of T_M:

$$\begin{aligned}\tau(a,\vec{T}) &\equiv \hat{\tau}(\hat{a}) \\ &= \tau(a,\vec{0}) + \frac{\partial\tau(a,\vec{0})}{\partial a}\sum_{M=1}^{\infty} T_M s_M(\hat{\tau}) + \frac{1}{2!}\frac{\partial^2\tau(a,\vec{0})}{\partial a^2}\sum_{M,N=1}^{\infty} T_M T_N s_M(\hat{\tau}) s_N(\hat{\tau}) + \cdots\,,\end{aligned} \tag{5.7}$$

where $s_M(\hat{\tau})$ should be also made explicit by iterating the expansion

$$\begin{aligned}s_M(\hat{\tau}) = s_M(\tau(a,0)) &+ \frac{\partial s_M(\tau(a,0))}{\partial a}\sum_{M=1}^{\infty} T_M s_M(\hat{\tau}) \\ &+ \frac{1}{2!}\frac{\partial^2 s_M(\tau(a,0))}{\partial a^2}\sum_{M,N=1}^{\infty} T_M T_N s_M(\hat{\tau}) s_N(\hat{\tau}) + \cdots\,.\end{aligned}$$

Thus we see that the initial function $\tau(a,\vec{0})$ is a generator of the hodograph solution (5.1).

With the replacement a by $\hat{a}$ (5.6) becomes

$$\hat{a} = \sum_{M=1}^{M_0} \Lambda_M s_M(\hat{\tau}(\hat{a}))\,. \tag{5.8}$$

Combining (5.3) and (5.8) we have

$$a + \sum_{M=1}^{M_0} \tilde{T}_M s_M(\tau(a,\vec{T})) = 0\,, \tag{5.9}$$

with $\tilde{T}_M = T_M - \Lambda_M$. This is a constraint satisfied by the hodograph solution, which was called the string equation in the topological field theory [13]. Multiplying (5.9) by $\partial\tau/\partial a$ and using the Whitham equation (4.5) gives

$$\left[a\frac{\partial}{\partial a} + \sum_{M=1}^{M_0} \tilde{T}_M \frac{\partial}{\partial T_M}\right]\tau = 0\,. \tag{5.10}$$

From this

$$\sum_{A=0}^{\infty} \tilde{T}_A \frac{\partial \tilde{\tau}_{BC}}{\partial T_A} = 0\,, \tag{5.11}$$

with $\tilde{T}_0 = a$, since $\tilde{\tau}_{BC}$ are functions of τ. By means of (5.11) we can easily show that the free energy in (4.10) takes the form

$$\mathcal{F} = \sum_{A,B=0}^{\infty} \frac{1}{2} \tilde{T}_A \tilde{T}_B \tilde{\tau}_{AB}(\tau(a, \vec{T}))\,, \tag{5.12}$$

modulo linear terms in T_A. Putting the hodograph solution (5.7) in (5.12) yields the free energy in a formal series of T_A. When $T_M = 0$, ${}^\forall M$, it becomes

$$\mathcal{F}_0(a) = \frac{1}{2} a^2 \tau((a, \vec{0})) - \sum_{M=1}^{M_0} a \Lambda_M \tau_{0M}(\tau(a, \vec{0})) + \frac{1}{2} \sum_{M,N=1}^{M_0} \Lambda_M \Lambda_N \tilde{\tau}_{MN}(\tau(a, \vec{0}))\,. \tag{5.13}$$

We finally note that the free energy (5.12) satisfies

$$2\mathcal{F} = \sum_{A=0}^{\infty} \tilde{T}_A \frac{\partial \mathcal{F}}{\partial T_A}\,, \tag{5.14}$$

and

$$\frac{\partial \mathcal{F}}{\partial T_B} = \sum_{A=0}^{\infty} \tilde{T}_A \tilde{\tau}_{AB}\,, \tag{5.15}$$

by (5.11).

In [15] the hodograph solution for the dispersionless KP hierarchy was interpreted in terms of the topological field theory. The same interpretation is applicable also for the case of the Whitham hierarchy. Namely the solution $\tau(a, \vec{T})$ and the corresponding free energy $\mathcal{F}$ are regarded as flowing in the *large phase space* of the scaling parameters T_M. On the other hand the initial values $\tau_0(a)$ and $\mathcal{F}_0(a)$ are regarded as flowing in the *small phase space* where the flow parameters $\tilde{T}_M(= T_M - \Lambda_M)$ are fixed to be $-\Lambda_M$. Such a *small phase space* may be associated with a Higgs vacuum in the quantum field theory, because $\tau_0(a)$ depends on the set of non-vanishing Λ_M and the hodograph solution is perturbatively constructed from $\tau_0(a)$ as (5.7). This aspect will be discussed in section 8 by using a matrix model.

6. The dual Whitham equation

In this section we discuss a dual version of the Whitham equation (4.5). Let us take a Legendre transform of the flow parameter a to a_D by

$$a_D = \frac{\partial \mathcal{F}}{\partial a}\,. \tag{6.1}$$

We then consider τ as a function of a_D and $\vec{T}$. By differentiating τ as

$$\frac{\partial \tau}{\partial T_M} = \frac{\partial \tau}{\partial a_D} \left(\frac{\partial a_D}{\partial T_M} \right)_{a=\text{const}} + \left(\frac{\partial \tau}{\partial T_M} \right)_{a_D=\text{const}},$$

the Whitham equation (4.5) becomes

$$\left(\frac{\partial \tau}{\partial T_M}\right)_{a_D=\text{const}} = s_{DM}(\tau)\frac{\partial \tau}{\partial a_D}\,, \tag{6.2}$$

with

$$s_{DM}(\tau) = \tau s_M(\tau) - \tau_{0M}\,. \tag{6.3}$$

Here use is made of (4.10) and

$$\frac{\partial a_D/\partial \tau}{\partial a/\partial \tau} = \frac{\partial a_D}{\partial a} = \tau\,. \tag{6.4}$$

The last equality follows from (6.1) and (4.10). $s_{DM}(\tau)$ is a dual version of the characteristic speed (3.5). With this form of $s_{DM}(\tau)$, (6.2) may be called the dual Whitham equation.

The Legendre transform (6.1) also induces the dual version of other equations. For instance, from (5.9) and (6.3) we have

$$a\tau + \sum_{M=1}^{\infty} \tilde{T}_M(s_{DM}(\tau) + \tau_{0M}) = 0\,,$$

which becomes

$$a_D + \sum_{M=1}^{\infty} \tilde{T}_M s_{DM}(\tau) = 0\,, \tag{6.5}$$

by (5.15). In the case when $T_M = 0$, $^\forall M$, (6.5) is reduced to

$$a_D = \sum_{M=1}^{M_0} \Lambda_M s_{DM}(\tau(a,\vec{0}))\,, \tag{6.6}$$

which is dual to the relation (5.6). They are paired as

$$a = \oint_A dS\,, \tag{6.7}$$

$$a_D = \oint_B dS\,, \tag{6.8}$$

with the differential dS

$$dS = \sum_{M=1}^{M_0} \Lambda_M[s_M(\tau)d\omega_0 - d\tilde{\Omega}_M]\,, \tag{6.9}$$

owing to (2.2), (2.10) and (4.4). By (6.3) we remark that

$$a_D = \tau a - \sum_{M=1}^{M_0} \Lambda_M \tau_{0M}\,, \tag{6.10}$$

in the case when $T_M = 0$, $^\forall M$. In the next section we will show that the Seiberg-Witten and Dijkgraaf-Vafa differentials are respectively obtained as special cases of the differential (6.9).

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The Legendre transform of the free energy $\mathcal{F}$ is given by

$$\mathcal{F}_D = aa_D - \mathcal{F} \,.$$

We can easily show that

$$\begin{aligned} &\frac{\partial^2 \mathcal{F}}{\partial a^2} = \tau \,, \\ &\frac{\partial \mathcal{F}_D}{\partial a_D} = a \,, \qquad \frac{\partial^2 \mathcal{F}_D}{\partial a_D^2} = \frac{1}{\tau} \,, \qquad \text{etc.} \end{aligned} \tag{6.11}$$

The same formulae hold for $\mathcal{F}_0$ and $\mathcal{F}_{0D}$. Particularly the following formulae are useful later:

$$\frac{\partial \mathcal{F}_0}{\partial a} = \oint_B dS \,, \tag{6.12}$$

$$\frac{\partial \mathcal{F}_0}{\partial(-\Lambda_M)} = -2\pi i \ \mathrm{res}_{\xi=0} \left[\frac{\xi^{-2M+1}}{2M-1} dS \right] , \tag{6.13}$$

which can be shown from (5.15) by means of (2.9)–(2.12) and (6.9).

We see that (6.7) with the differential dS (6.9) is another expression of the initial constraint (5.6), which was inverted to give the initial condition $\tau_0(a)$ for the hodograph solution. Choosing a different set of non-vanishing Λ_M specifies the differential dS. Correspondingly the free energy $\mathcal{F}_0$ is fixed according to (5.13). When the flow parameters T_M, $M = 1, 2, \ldots$, are turned on, $\mathcal{F}_0$ flows in the *large phase space* as given by $\mathcal{F}$ with (5.7).

7. Applications

7.1 $N = 2$ effective Yang-Mills theory with $\mathrm{SU}(2)$

The relevant curve takes the form

$$y^2 = 4(x^2 - 1)(x - u) \,.$$

We put it in the Weierstrass standard form (A.1) with

$$e_1 = \frac{2}{3}u \,, \qquad e_2 = 1 - \frac{u}{3} \,, \qquad e_3 = -1 - \frac{u}{3} \,.$$

A simple manipulation of (A.2) gives the relations

$$u = -1 + 2 \left[\frac{\theta_3(\tau)}{\theta_2(\tau)} \right]^4 , \tag{7.1}$$

$$g_0 = -2\omega_1 = \frac{\pi}{\sqrt{2}} \theta_2(\tau)^2 \,. \tag{7.2}$$

We consider the hodograph solution of the Whitham equation (4.5), imposing the initial condition (5.6) with $\Lambda_M = 0$ for $M \geq 2$, i.e.,

$$a = \Lambda_1 s_1(\tau) \,. \tag{7.3}$$

Here $s_1(\tau)$, given by (3.5), can be calculated as

$$s_1(\tau) = g_0 Q_1(u) = g_0(u + \gamma_0)\,,$$

with

$$\gamma_0 = -\frac{1}{g_0}\oint_A x\frac{dx}{y}\,.$$

The period integral is evaluated in the Weierstrass standard form as

$$\gamma_0 = -\frac{1}{g_0}\oint_A \left(t + \frac{u}{3}\right)\frac{dt}{y} = \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 E_2(\tau) - \frac{u}{3}\,.$$

According to (6.7)–(6.9) the initial condition (7.3) can be put in the form

$$a = -\Lambda_1 \oint_A (x-u)\frac{dx}{y} \tag{7.4}$$

in which the integrand is the Seiberg-Witten differential dS. The free energies (5.12) and (5.13) are given with all Λ_M vanishing except Λ_1, i.e.,

$$\mathcal{F} = \mathcal{F}|_{\tilde{T}_1 = T_1 - \Lambda_1, \tilde{T}_{M\geq 2} = T_{M\geq 2}}\,, \tag{7.5}$$

and

$$\begin{aligned}\mathcal{F}_0(a) &= \mathcal{F}|_{\text{all } T_M=0} \\ &= \frac{1}{2}a^2\tau(a,\vec{0}) - a\Lambda_1\tau_{01}(\tau(a,\vec{0})) + \frac{1}{2}\Lambda_1^2\tau_{11}(\tau(a,\vec{0}))\,.\end{aligned} \tag{7.6}$$

Here note that $\tilde{\tau}_{MN}$ is reduced to τ_{MN} because $\Delta_M(x,\tau) = 0$. To get explicit forms of the free energies (7.5) and (7.6) we need to calculate all of τ_{MN} as in appendix B. With the free energy (7.6) the effective lagrangian of the $N = 2$ Yang-Mills theory with SU(2) is given by [5]

$$W_{SW} = \frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\, \frac{\partial\mathcal{F}_0(A)}{\partial A}\bar{A} + \int d^2\theta\, \frac{1}{2}\tau(A,\vec{0})W_\alpha W^\alpha\right],$$

in which A and W_α are $N = 1$ U(1) chiral and vector superfields respectively.

7.2 $N = 1^*$ theory

The $N = 1^*$ theory with U(N) is characterized by the effective superpotential [2, 12]

$$W_{\text{eff}} = \int d^2\theta\, (N a_D - \mathring{\tau} a)\,. \tag{7.7}$$

Here a and a_D are given by

$$a = \frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau))\,, \tag{7.8}$$

$$a_D = \tau\frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau)) + \frac{i}{12}E_2(\tau) = \frac{\partial\mathcal{F}_0}{\partial a}\,, \tag{7.9}$$

which were denoted by $\Pi_A(=2\pi i S)$ and Π_B respectively in [2, 12]. It is important to remark that $\partial a_D/\partial a=\tau$ is guaranteed by the formula

$$\frac{\partial}{\partial\tau}E_2(\tau)=\frac{i\pi}{6}(E_2(\tau)^2-E_4(\tau))\,. \tag{7.10}$$

The bare coupling $\overset{\circ}{\tau}(\equiv\theta/2\pi+4\pi/g_{\rm eff}^2)$ is related by extremizing the superpotential as [2]

$$\tau=\frac{\overset{\circ}{\tau}+k}{N}\,,\qquad k=0,1,2,\ldots,N-1\,.$$

We shall find an explicit form of the free energy $\mathcal{F}_0$ in (7.9). To this end, we shall put the $N=1^*$ theory in the formalism developed in sections 5 and 6. Note the similarity between the set of the equations (7.8)–(7.10) and that of (5.6), (6.10) and

$$\frac{\partial}{\partial\tau}\sum_{M=1}^{M_0}\Lambda_M\tau_{0M}=\sum_{M=1}^{M_0}\Lambda_M s_M(\tau)\,. \tag{7.11}$$

The last equation is due to (4.5) and(4.6). Therefore if a, given by (7.8), is identified as

$$\frac{\pi}{72}(E_2(\tau)^2-E_4(\tau))=\sum_{M=1}^{M_0}\Lambda_M s_M(\tau)\,, \tag{7.12}$$

with certain parameters $(\Lambda_1,\Lambda_2,\ldots,\Lambda_{M_0})$, we have in (7.9)

$$\frac{i}{12}E_2(\tau)=-\sum_{M=1}^{M_0}\Lambda_M\tau_{0M}\,, \tag{7.13}$$

and *vice versa.* (7.8) can be considered as the constraint which gives by inversion the initial condition $\tau_0(a)$ for the hodograph solution. Then the identification (7.12) or (7.13) implies that the $N=1^*$ theory can be characterized by the differential dS, (6.9) and correspondingly the free energy $\mathcal{F}_0$, (5.13). In the present case they read respectively

$$dS=-\sum_{M=1}^{M_0}\Lambda_M d\tilde{\Omega}_M+\frac{\pi}{72}(E_2(\tau)^2-E_4(\tau))d\omega_0\,, \tag{7.14}$$

and

$$\mathcal{F}_0(a)=\frac{1}{2}a^2\tau(a,\vec{0})+\frac{i}{12}aE_2(\tau(a,\vec{0}))+\frac{1}{2}\sum_{M,N=1}^{M_0}\Lambda_M\Lambda_N\tilde{\tau}_{MN}(\tau(a,\vec{0}))\,. \tag{7.15}$$

Calculating $Q_M(x)$ and $\tilde{\tau}_{MN}$ by using appendix B we find a concrete form of the free energy $\mathcal{F}_0$ in (7.9). It is important remark that this free energy remains the same independently of which set to take for $(\Lambda_1,\Lambda_2,\ldots,\Lambda_{M_0})$. It is due to (7.12) or equivalently (7.13), as shown by the following calculation

$$\frac{\partial}{\partial\tau}\sum_{M,N=1}^{M_0}\Lambda_M\Lambda_N\tilde{\tau}_{MN}\;=\;\sum_{M,N=1}^{M_0}\Lambda_M\Lambda_N\left[\frac{\partial}{\partial\tau}\tau_{MN}+2\pi i\,{\rm res}_{\xi=0}[\xi^{-2M}d\xi\Delta_N(x,\tau)]\right]$$

$$= -2\pi i \sum_{M,N=1}^{M_0} \Lambda_M \Lambda_N \mathrm{res}_{\xi=0} \left[\frac{\xi^{-2M+1}}{2M-1} s_N(\tau) \frac{\partial}{\partial\tau} d\omega_0 \right]$$

$$= \left[\sum_{M=1}^{M_0} \Lambda_M s_M(\tau) \right]^2 = \left[\frac{\pi}{72} (E_2(\tau)^2 - E_4(\tau)) \right]^2 , \tag{7.16}$$

with the help of (2.10), (3.4), (4.9) and (7.11). In other words, depending on the choice of $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{M_0})$ we are here changing the characteristic speeds $s_M(\tau)$, i.e., parameterization of the curve, so as to satisfy (7.12) or equivalently (7.13). On the contrary, in section 5 we have done in the opposite way, i.e., we kept the characteristic speeds the same, but changed the initial condition $\tau_0(a)$ by choosing $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{M_0})$ differently.

To see this, we closely look into the constraint (7.13). By using the formulae (B.8) for τ_{0M} it reads

$$\Lambda_1 = -\frac{1}{24} \frac{g_0}{\pi} E_2(\tau) , \qquad \text{for } M_0 = 1 , \tag{7.17}$$

$$\Lambda_2 c + 6\Lambda_1 = -\frac{1}{4} \frac{g_0}{\pi} E_2(\tau) , \qquad \text{for } M_0 = 2 , \tag{7.18}$$

$$\Lambda_3 c^2 + 4\Lambda_2 c + \frac{3}{5} \Lambda_3 g_2 + 24\Lambda_1 = -\frac{g_0}{\pi} E_2(\tau) , \qquad \text{for } M_0 = 3 , \tag{7.19}$$

and so on.

i) case with $M_0 = 1$: it suffices to consider the constraint (7.17) as the "gauge" condition for g_0, discussed in section 3. The differential (7.14) takes the form

$$dS = -\Lambda_1 \left[\left(x - \frac{c}{3} - g_1 \right) \frac{dx}{y} - d \left(\int^\tau d\tau \Delta_1(x, \tau) \right) \right] + \frac{\pi}{72} (E_2(\tau)^2 - E_4(\tau)) d\omega_0 , \tag{7.20}$$

by determining $Q_1(x)$ by the requirement (2.7). The corresponding free energy (7.15) becomes

$$\mathcal{F}_0(a) = \frac{1}{2} a^2 \tau(a, \vec{0}) + \frac{i}{12} a E_2(\tau(a, \vec{0})) + \frac{1}{2} \Lambda_1^2 \tilde{\tau}_{11}(\tau(a, \vec{0})) , \tag{7.21}$$

with $\tilde{\tau}_{11}$ defined by (4.9), i.e.,

$$\tilde{\tau}_{11} = 2\pi i \left[\frac{c}{6} - g_1 + \int^\tau d\tau \ \mathrm{res}_{\xi=0} [\xi^{-2} d\xi \Delta_1(x, \tau)] \right] . \tag{7.22}$$

Here τ_{11} was calculated by (B.8).

ii) case with $M_0 = 2$: for simplicity we set $\Lambda_1 = 0$. Then (7.18) requires c to be

$$c = -\frac{1}{4\Lambda_2} \frac{g_0}{\pi} E_2(\tau) . \tag{7.23}$$

Here g_0 is still to be fixed as the "gauge" condition. With c constrained as such, the differential (7.14) and the free energy (7.15) are respectively given by

$$dS = -\Lambda_2 \left[\left(x^2 - \frac{c}{2} x + \frac{c^2}{18} - \frac{c}{6} g_1 - \frac{g_2}{12} \right) \frac{dx}{y} - d \left(\int^\tau d\tau \Delta_2(x, \tau) \right) \right] +$$
$$+ \frac{\pi}{72} (E_2(\tau)^2 - E_4(\tau)) d\omega_0 , \tag{7.24}$$

and

$$\mathcal{F}_0(a) = \frac{1}{2}a^2\tau(a,\vec{0}) + \frac{i}{12}aE_2(\tau(a,\vec{0})) + \frac{1}{2}\Lambda_2^2\tilde{\tau}_{22}(\tau(a,\vec{0}))\,, \tag{7.25}$$

with

$$\tilde{\tau}_{22} = \pi i\left[\frac{c^3}{324} - \frac{c^2}{18}g_1 + \frac{c}{36}g_2 + \frac{1}{12}g_3 + 2\int^{\tau} d\tau\ \mathrm{res}_{\xi=0}[\xi^{-4}d\xi\Delta_2(x,\tau)]\right]. \tag{7.26}$$

Here $Q_2(x)$ in dS was calculated by means of (B.3) and (B.6), while $\tilde{\tau}_{22}$ in (B.8).

iii) case with $M_0 = 3$: for simplicity set $\Lambda_2 = \Lambda_1 = 0$. Then (7.19) constrains c such that

$$c^2 = -\frac{1}{\Lambda_3}\left[\frac{g_0}{\pi}E_2(\tau) + \frac{3}{5}g_2\right]. \tag{7.27}$$

Here also the "gauge" freedom g_0 is still to be fixed. Similarly to the previous case we calculate the differential (7.14) and the free energy (7.15) by the formulae in appendix B. Then with c given by (7.27) they are found to take the respective forms

$$\begin{aligned} dS \;=\; -\Lambda_3\Bigg[&\left\{x^3 - \frac{c}{2}x^2 + \left(\frac{c^2}{24} - \frac{g_2}{8}\right)x + \frac{c^3}{216} - \frac{c^2}{24}g_1 - \frac{g_3}{10} - \frac{g_1g_2}{40}\right\}\frac{dx}{y} - \\ &- d\left(\int^{\tau} d\tau\Delta_3(x,\tau)\right)\Bigg] + \frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau))d\omega_0\,, \end{aligned} \tag{7.28}$$

and

$$\mathcal{F}_0(a) = \frac{1}{2}a^2\tau(a,\vec{0}) + \frac{i}{12}aE_2(\tau(a,\vec{0})) + \frac{1}{2}\Lambda_3^2\tilde{\tau}_{33}(\tau(a,\vec{0}))\,, \tag{7.29}$$

with

$$\begin{aligned} \tilde{\tau}_{33} \;=\; \pi i\Bigg[&\frac{c^5}{8640} - \frac{c^4}{288}g_1 + \frac{c^3}{288}g_2 + \frac{c^2}{240}(-g_2g_1 + 6g_3) + \frac{c}{192}(g_2)^2 + \\ &+ \frac{3}{400}g_3g_2 - \frac{1}{800}(g_2)^2g_1 + 2\int^{\tau} d\tau\ \mathrm{res}_{\xi=0}[\xi^{-6}d\xi\Delta_3(x,\tau)]\Bigg]. \end{aligned} \tag{7.30}$$

So far there remains the "gauge" freedom in both cases ii) and iii). For the case ii) let us take the "gauge" $g_0 = 2\pi$ and set $\Lambda_2 = 1$. Then the constraint (7.23) becomes $c = -\frac{1}{2}E_2(\tau)$. The Dijkgraaf-Vafa differential (7.24) is simplified as

$$dS = -\left[t^2 - \frac{1}{12}E_2(\tau)t - \frac{1}{72}E_2(\tau)^2\right]\frac{dt}{y} + d\left(\int^{\tau} d\tau\Delta_2(x,\tau)\right). \tag{7.31}$$

with $t = x + \frac{1}{6}E_2(\tau)$. This is the Dijkgraaf-Vafa differential given in [2]. The free energy is reduced to

$$\begin{aligned} \mathcal{F}_0(a) \;=\; &\frac{1}{2}a^2\tau(a,\vec{0}) + \frac{i}{12}aE_2(\tau(a,\vec{0})) + \\ &+\frac{\pi i}{2}\left[\frac{1}{1296}E_2(\tau(a,\vec{0}))^3 - \frac{1}{864}E_2(\tau(a,\vec{0}))E_4(\tau(a,\vec{0})) + \frac{1}{2592}E_6(\tau(a,\vec{0}))\right] + \\ &+\pi i\int^{\tau} d\tau\ \mathrm{res}_{\xi=0}[\xi^{-4}d\xi\Delta_2(x,\tau)]\,. \end{aligned}$$

By plugging a given by (7.8) and evaluating the last term as (C.22) in appendix C it takes the form

$$
\begin{aligned}
\mathcal{F}_0(a) = & \frac{1}{2}\left(\frac{\pi}{72}\right)^2 (E_2(\tau)^2 - E_4(\tau))^2\tau + \frac{\pi i}{72^2}(8E_2(\tau)^3 - 9E_2(\tau)E_4(\tau) + E_6(\tau)] + \\
& + \frac{1}{2}\left(\frac{\pi}{72}\right)^2 \int^\tau d\tau[6E_2(\tau)E_6(\tau) - 9E_2^2(\tau)E_4(\tau) + 3E_2(\tau)^4]\,,
\end{aligned}
$$

which may be now considered as a function of τ.

Thus we have shown that the $N = 1^*$ theory can be characterized by the different DV differentials (7.20), (7.24) and (7.28). But the corresponding free energies (7.21), (7.25) and (7.29) are all kept the same, as has been proved by the calculation (7.16). Nonetheless they look quite different. As a consistency check of the whole formalism, it is worth showing directly that

$$
\Lambda_1^2 \frac{\partial}{\partial\tau}\tilde{\tau}_{11} = \Lambda_2^2 \frac{\partial}{\partial\tau}\tilde{\tau}_{22} = \Lambda_3^2 \frac{\partial}{\partial\tau}\tilde{\tau}_{33} = \left[\frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau))\right]^2 \qquad (7.32)
$$

for $\tilde{\tau}_{11}, \tilde{\tau}_{22}$ and $\tilde{\tau}_{33}$ given by (7.22), (7.26) and (7.30). It will be done in appendix C.

But difference due to the respective characterization of the $N = 1^*$ theory by (7.20), (7.24) and (7.28) appears when we discuss the Whitham deformation of the theory. The Whitham deformation is governed by the hodograph solution (5.7). The curve is parameterized in such a way that for each case of i), ii) and iii) one of the characteristic speeds $s_M(\tau)$, $M = 1, 2, \ldots$, takes the fixed functional form, *i.e.*,

$$
\left.\begin{array}{ll}
\Lambda_1 s_1(\tau) & (\text{ of case i})) \\
\Lambda_2 s_2(\tau) & (\text{ of case ii})) \\
\Lambda_3 s_3(\tau) & (\text{ of case iii}))
\end{array}\right\} = \frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau))\,. \qquad (7.33)
$$

Then all other $s_M(\tau)$ differ depending on the case to be considered. Hence the hodograph solution (5.7) flows differently from the same initial condition $\tau_0(a)$. So does the free energy $\mathcal{F}_0(a)$, as given by (5.12). (See figure 1.) In other words, the curve (2.1) is parameterized differently for each case of i), ii) and iii). The branch points are given by (A.2) as

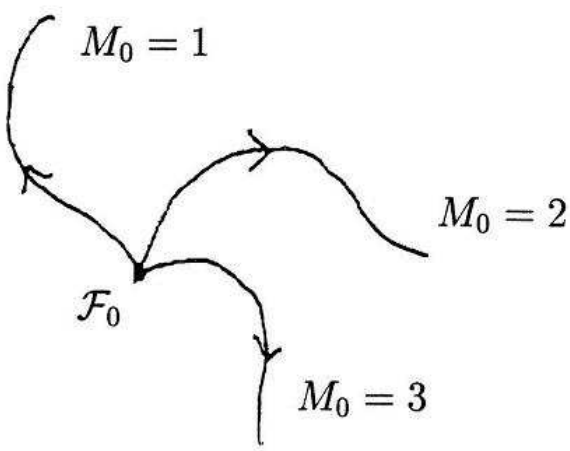

Figure 1: Flows of the free energy $\mathcal{F}_0$.

$$
\begin{aligned}
u(\tau) &= \frac{c}{3} + \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 [\theta_3(\tau)^4 + \theta_0(\tau)^4]\,, \\
v(\tau) &= \frac{c}{3} + \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 [\theta_2(\tau)^4 - \theta_0(\tau)^4]\,, \\
w(\tau) &= \frac{c}{3} + \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 [\theta_2(\tau)^4 + \theta_3(\tau)^4]\,,
\end{aligned} \qquad (7.34)
$$

in which c as well as g_0 are different depending on the case. (g_0 is the "gauge" freedom to be fixed arbitrarily, except for case i)). Then these branch points move through deformation of τ as given by the hodograpgh solution (5.7). The curve might happen to degenerate at some flow points $\vec{T} \neq 0$. We may expect a large variety of degeneracy of the $N = 1^*$ theory, depending on the choice of c, though the free energy is initially the same.

8. Interpretation of the flow parameters Λ_M

We shall give a physical interpretation of the flow parameters Λ_M which are so far mathematical. Namely we will show that the free energy (5.13) in the *small phase space* is equivalent to the one of the matrix model given by

$$Z = \int d\Phi \frac{e^{-NtrV(\Phi)}}{\det([\Phi, -] + i)}, \tag{8.1}$$

with

$$V(\Phi) = \sum_{M=0}^{M_0} \bar{\Lambda}_M \Phi^{2M}.$$

Here Φ is an $N \times N$ hermite matrix and $\bar{\Lambda}_M$ is a linear combination of Λ_M which will be given later. It can be done by making use of the arguments in [3]. Namely they discussed the relation between the Toda hierarchy and a matrix model. The matrix model (8.1) is its variant and was studied in [2, 12, 16]. There underlies the KdV hierarchy. So we shall adapt the arguments in [3] to this case. By diagonalizing Φ the integral (8.1) is reduced to an integral over the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then we obtain the saddle point equation

$$-NV'(\lambda_I) + \sum_{J \neq I} \left[\frac{2}{\lambda_I - \lambda_J} - \frac{1}{\lambda_I - \lambda_J + i} - \frac{1}{\lambda_I - \lambda_J - i} \right] = 0. \tag{8.2}$$

Let us take the large-N limit and assume that the eigenvalues spread out along the real line $[-\alpha, \alpha]$ with the density $\rho(\lambda)$. The integral (8.1) is written in the form

$$\begin{aligned} \log Z \; = \; -N^2 \Big\{ \int_{-\alpha}^{\alpha} d\lambda \rho(\lambda) V(\lambda) - \frac{1}{2} \int_{-\alpha}^{\alpha} d\lambda \rho(\lambda) \int_{-\alpha}^{\alpha} d\lambda' \rho(\lambda') \times \\ \times \left[2 \log(\lambda - \lambda') - \log(\lambda - \lambda' + i) - \log(\lambda - \lambda' - i) \right] \Big\}. \end{aligned} \tag{8.3}$$

and (8.2) becomes

$$V'(\lambda) - \int_{-\alpha}^{\alpha} d\lambda' \rho(\lambda') \left[\frac{2}{\lambda - \lambda'} - \frac{1}{\lambda - \lambda' + i} - \frac{1}{\lambda - \lambda' - i} \right] = 0, \tag{8.4}$$

for $\lambda \in [-\alpha, \alpha]$. The l.h.s. of this equation is a force acting on a test eigenvalue λ. Introducing the resolvent

$$\omega(\lambda) = \int_{-\alpha}^{\alpha} d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'},$$

we rewrite this force as

$$f(\lambda) \equiv V'(\lambda) - [2\omega(\lambda) - \omega(\lambda + i) - \omega(\lambda - i)]. \tag{8.5}$$

Then the saddle point equation (8.4) reads

$$f(\lambda) = 0\,, \qquad \text{for } \lambda \in [-\alpha, \alpha]\,. \tag{8.6}$$

We further introduce the function

$$G(\lambda) = \mathrm{U}(\lambda) + i\left[\omega\left(\lambda + \frac{i}{2}\right) - \omega\left(\lambda - \frac{i}{2}\right)\right], \tag{8.7}$$

with a polynomial $\mathrm{U}(\lambda)$ such that

$$V'(\lambda) = -i\left[\mathrm{U}\left(\lambda + \frac{i}{2}\right) - \mathrm{U}\left(\lambda - \frac{i}{2}\right)\right]. \tag{8.8}$$

To be concrete, $\mathrm{U}(\lambda)$ is found in the form

$$\frac{\partial\,\mathrm{U}(\lambda)}{\partial\bar{\Lambda}_M} = \sum_{k=0}^{M} c_k^{(M)}\lambda^{2k}\,, \qquad c_M^{(M)} = 1\,, \tag{8.9}$$

in which $c_k^{(M)}$ are numerical values recursively determined with the requirement (8.8). In terms of the function $G(\lambda)$ the saddle point equation (8.6) becomes

$$G(\lambda + \frac{i}{2}) = G\left(\lambda - \frac{i}{2}\right), \qquad \text{for } \lambda \in [-\alpha, \alpha]\,. \tag{8.10}$$

This function has two branch cuts along

$$\mathcal{C}_+ = \left[-\alpha + \frac{i}{2}, \alpha + \frac{i}{2}\right], \qquad \mathcal{C}_- = \left[-\alpha - \frac{i}{2}, \alpha - \frac{i}{2}\right].$$

The saddle point equation (8.10) implies that $G(\lambda)d\lambda$ is a meromorphic differential on an elliptic curve with the pole structure read in (8.7). We take A-cycle and the dual B-cycle in the same way as in [2]. We also define the function $\xi(\lambda)$ by integrating the force (8.5):

$$\begin{aligned}\xi(\lambda) \equiv&\int_\infty^\lambda d\lambda f(\lambda)\\ =&\, V(\lambda) - \int_{-\alpha}^{\alpha} d\lambda'\rho(\lambda')[2\log(\lambda - \lambda') - \log(\lambda - \lambda' + i) - \log(\lambda - \lambda' - i)]\,.\end{aligned} \tag{8.11}$$

It is known [2] that for $\lambda \in [-\alpha, \alpha]$ this function is λ-independent as

$$\xi(\lambda) = \int_\infty^\alpha d\lambda f(\lambda) = i\oint_B d\lambda G(\lambda)\,. \tag{8.12}$$

Noting that the jump in $G(\lambda)$ along the branch cut $\mathcal{C}_+$ (going upwards) is $2\pi\rho(\lambda)$, we rewrite the free energy (8.3) as

$$\begin{aligned}\log Z =& -N^2\left\{\frac{1}{2}\oint_A \frac{d\lambda}{2\pi}G(\lambda)V(\lambda) - \frac{1}{8}\oint_A\frac{d\lambda}{2\pi}\oint_A\frac{d\lambda'}{2\pi}G(\lambda)G(\lambda')\,\times\right.\\ &\left.\times\,[2\log(\lambda - \lambda') - \log(\lambda - \lambda' + i) - \log(\lambda - \lambda' - i)]\right\}.\end{aligned} \tag{8.13}$$

Let us define the variable $\bar{a}$ by

$$\bar{a} = \oint_A G(\lambda) d\lambda\,. \tag{8.14}$$

We differentiate the free energy (8.13) by this $\bar{a}$:

$$\frac{\partial \log Z}{\partial \bar{a}} = -\frac{N^2}{2}\oint_A \frac{d\lambda}{2\pi}\frac{\partial G(\lambda)}{\partial \bar{a}}\xi(\lambda)\,.$$

Remember the fact that $\xi(\lambda)$ is λ-independent on the A-cycle and written as (8.12). Then this becomes

$$\begin{aligned}\frac{\partial \log Z}{\partial \bar{a}} &= -i\frac{N^2}{2}\oint_A \frac{d\lambda}{2\pi}\frac{\partial G(\lambda)}{\partial \bar{a}}\oint_B d\lambda G(\lambda) \\ &= -i\frac{N^2}{4\pi}\oint_B d\lambda G(\lambda)\,,\end{aligned} \tag{8.15}$$

due to (8.14). By a similar calculation we have

$$\frac{\partial \log Z}{\partial \bar{\Lambda}_M} = -N^2\left[\frac{1}{2}\oint_A \frac{d\lambda}{2\pi}\frac{\partial G(\lambda)}{\partial \bar{\Lambda}_M}\xi(\lambda) + \frac{1}{2}\oint_A \frac{d\lambda}{2\pi}G(\lambda)\frac{\partial V(\lambda)}{\partial \bar{\Lambda}_M}\right]. \tag{8.16}$$

Use $\partial\bar{a}/\partial\bar{\Lambda}_M = 0$ and the properties of $\xi(\lambda)$ again. It may be written as

$$\frac{\partial \log Z}{\partial \bar{\Lambda}_M} = -i\frac{N^2}{2}\oint_A \frac{d\lambda}{2\pi}G(\lambda)\oint_B d\lambda\frac{\partial G(\lambda)}{\partial \bar{\Lambda}_M}\,. \tag{8.17}$$

Finally we have recourse to the Riemann bilinear formula to proceed the calculation. Knowing the singular behavior of $\partial G(\lambda)/\partial\bar{\Lambda}_M$ from (8.9) we obtain

$$\frac{\partial \log Z}{\partial \bar{\Lambda}_M} = i\frac{N^2}{4\pi}\sum_{k=0}^{M} 2\pi i\; c_k^{(M)}\mathrm{res}_{\lambda=\infty}\left[\frac{\lambda^{2k+1}}{2k+1}G(\lambda)d\lambda\right]. \tag{8.18}$$

We think of the mapping from the λ-plane to the z-plane parameterizing the torus with the period ω_2/ω_1. In [16] it was given by

$$\lambda(z) = -\frac{\omega_1}{\pi}\left[\zeta(z) - \frac{\zeta(\omega_1)}{\omega_1}z\right] = -\frac{\omega_1}{\pi}\left[\frac{1}{z} + O(z)\right]. \tag{8.19}$$

with $\zeta(z)$ the Weierstrass ζ-function. By using this coordinate z (8.18) becomes

$$\frac{\partial \log Z}{\partial \bar{\Lambda}_M} = -i\frac{N^2}{4\pi}\sum_{k=0}^{M} 2\pi i\, c_k^{(M)}\left(\frac{\omega_1}{\pi}\right)^{2k+1}\mathrm{res}_{z=0}\left[\frac{z^{-2k-1}}{2k+1}G(\lambda)d\lambda\right]. \tag{8.20}$$

If we can identify the meromorphic differential $G(\lambda)d\lambda$ with dS and $\log Z$ with $-i\mathcal{F}_0$, then (8.15) and (8.18) become the equations (6.12) and (6.13) by appropriate linear combination of $\bar{\Lambda}_M$. Thus we can interpret the flow parameters Λ_M of the Whitham deformation as the coupling constants $\bar{\Lambda}_M$ of the matrix model (8.1). So we are left with the task to show $G(\lambda)d\lambda$ to take the same form as (6.9). To this end we write the elliptic curve (2.1) in the form

$$y^2 = 4t^3 - g_2\, t - g_3\,, \tag{8.21}$$

and note that it is mapped to the z-plane through the Weierstrass $\mathcal{P}$-function

$$t = -\zeta'(z) = \mathcal{P}(z)\,, \qquad y = \mathcal{P}'(z)\,. \tag{8.22}$$

Following [16] we shall determine the elliptic function $G(\lambda)$. From (8.7), (8.9) and (8.19) we know the pole singularity

$$[G(\lambda(z))]_- = \sum_{M=0}^{M_0} \bar{\Lambda}_M \sum_{k=0}^{M} c_k^{(M)} \left(\frac{\omega_1}{\pi}\right)^{2k} \sum_{i=0}^{k} d_i^{(k)} z^{-2i}\,, \tag{8.23}$$

in which $[\cdots]_-$ indicates the part of non-positive powers in z and $d_i^{(k)}$ are calculable constants with $d_k^{(k)} = 1$. By using the expansion

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \cdots\,, \tag{8.24}$$

it may be expressed by a polynomial of $\mathcal{P}(z)$ That is,

$$\begin{aligned} [G(\lambda(z))]_- &= \sum_{M=0}^{M_0} \bar{\Lambda}_M \sum_{k=0}^{M} c_k^{(M)} \left(\frac{\omega_1}{\pi}\right)^{2k} [\mathcal{P}(z)^k + d_{k-1}^{(k)} \mathcal{P}(z)^{k-1} + \cdots]_- \\ &\equiv \left[\sum_{M=0}^{M_0} \bar{\Lambda}_M \sum_{k=0}^{M} c_k^{(M)} \left(\frac{\omega_1}{\pi}\right)^{2k} \bar{P}_k(\mathcal{P}(z))\right]_-\,, \end{aligned} \tag{8.25}$$

with a polynomial $\bar{P}_k(\cdot)$ of degree k. Since an elliptic function is determined uniquely by the pole singularity, so that

$$G(\lambda(z)) = \sum_{M=0}^{M_0} \bar{\Lambda}_M \sum_{k=0}^{M} c_k^{(M)} \left(\frac{\omega_1}{\pi}\right)^{2k} \bar{P}_k(\mathcal{P}(z))\,. \tag{8.26}$$

By means of (8.19), (8.22) and (8.26) the meromorphic differential $G(\lambda)d\lambda$ can be written as

$$G(\lambda)d\lambda = \sum_{M=0}^{M_0} \bar{\Lambda}_M \sum_{k=0}^{M} c_k^{(M)} \left(\frac{\omega_1}{\pi}\right)^{2k+1} \left[t + \frac{\zeta(\omega_1)}{\omega_1}\right] \bar{P}_k(t) \frac{dt}{y}\,. \tag{8.27}$$

By a linear transformation from $\bar{\Lambda}_M$ to Λ_M this takes the form (6.9), except for the boundary term in $d\tilde{\Omega}_M$ which vanishes at $t = \infty$. But, when we interpreted the saddle point equation (8.10), we could allow $G(\lambda)d\lambda$ to have a boundary term like $d(\cdots)$ which is not meromorphic. As long as it vanishes at $\lambda = \infty$, the argument thereafter goes through without any modification. Appropriately fixing this freedom would yield the boundary term in dS. Or note that this boundary term depends on the gauge fixing discussed in section 3. Then we can simply say that $\log Z$ of the matrix model corresponds to the free energy $\mathcal{F}_0$ of the Whitham hierarchy calculated with the particular gauge in which the boundary term disappears, for instance, $v = const$ and $w = const$.

Thus we were able to identify the free energy (5.13) in the small phase space with the one of the matrix model (8.1). The free energy (5.12) in the *large phase space* may be identified as the one obtained by peturbing the model (8.1):

$$Z = \int d\Phi \frac{e^{-Ntr[\sum_{M=0}^{M_0} \bar{\Lambda}_M \Phi^{2M} - \sum_{M=0}^{\infty} \bar{T}_M \Phi^{2M}]}}{\det([\Phi, -] + i)}\,. \tag{8.28}$$

In section 5 we have shown how to obtain the free energy (5.12) by perturbing the one (5.13) in the *small phase space*. We can obtain the free energy of the model (8.28) by perturbing the model (8.1) exactly in the same sense. Thus the flow parameters T_M, $M = 1, 2, \ldots$ of the free energy (5.12) in the *large phase space* can be interpreted as the coupling constants $\bar{T}_M$ of the perturbative interaction of the matrix model (8.28).

The different characterizations of the free energy by (7.20), (7.24) and (7.28) in the previous section corresponds to the matrix model (8.1) with $M_0 = 0, 1, 2$ after an appropriate setting of $\bar{\Lambda}_M$. The three flows illustrated in figure 1 are interpreted as different perturbations of these matrix models by (8.28). Then (7.32) or more concretely (7.33) is interpreted as a condition which sets the free energy $\log Z$ with $M_0 = 0, 1, 2$ to be equal at $\bar{T}_M = 0$. It can be done by choosing the period $\tau = \omega_2/\omega_1$ of the torus appropriately.

We have discussed only the *two*-cut solution of the matrix model (8.1). We may consider multi-cut solutions on a Riemann surface with higher genus. It would be interesting to extend the whole arguments in this section to such general cases.

9. Conclusions

In this paper we have discussed the Whitham deformation of the free energy $\mathcal{F}_0$ which appears in the DV and SW theories. It amounts to discussing deformation of a relevant (hyper)elliptic curve with flow parameters. We were mainly concerned about the elliptic case. Generalization to the hyperelliptic case would be straightforward albeit with some technical complications. We then derived the Whitham equation for the period τ. Its hodograph solution represents the Whitham deformation of the free energy $\mathcal{F}_0$ in the *large phase space* of flow parameters. To find a hodograph solution we have to impose a constraint which determines an initial functional form $\tau_0(a)$ or $a(\tau_0)(\equiv \tau_0^{-1}(a))$. It amounts to determining the free energy $\mathcal{F}_0$ in the *small phase space*. The main message of this paper is that $\tau_0(a)$ and $\mathcal{F}_0$, given at one point in the *small phase space*, get deformed along different flows in the *large phase space*, when the elliptic curve is parameterized differently at that point. As an application of this argument we took the effective superpotential of the $N = 1^*$ theory (7.7). We have shown that the same superpotential can be indeed characterized by the DV differentials (7.20), (7.24) and (7.28) on different curves. For each chosen DV differential the superpotential undergoes different Whitham deformations. We have also given an interpretation of these Whitham deformations in terms of the matrix model.

The free energy $\mathcal{F}_0$ of the $N = 1^*$ theory took rather complicated forms for each case of i), ii) and iii). We emphasize that the term $\int^\tau d\tau(\cdots)$ coming from the boundary term in (3.4) is essential to have the property $\tau = \partial^2\mathcal{F}_0/\partial a^2$. If two of the branch points of the curve are fixed to be constant, for instance, $v = 1$ and $w = -1$ as in the $N = 2$ effective Yang-Mills theory in subsection 7.1, there is no contribution from the boundary. In such cases the free energy $\mathcal{F}_0$ is rather simply calculated. But in general the boundary term was necessary for the compatibility of the Whitham deformation (4.2) and (4.3).

The arguments in this paper can be applied for the SW theory in which the constraint (7.4) is inverted to give the initial condition

$$\tau_0(a) = \frac{g_{D0}}{g_0} \sim \text{const.} + \frac{i}{\pi}\log a^2\,, \qquad a \sim \infty\,, \tag{9.1}$$

by using (7.1) and (7.2). This condition is essentially related to the effective coupling g_{eff} of QCD. We might parameterize the elliptic curve in the general form (4.1) and realize the initial constraint (9.1) by the generalized differential (6.9) with $M_0 \geq 2$. Then a is expressed by the branch pints u, v and w. They move obeying the Whitham equation. We might think of the Whitham equation as a master equation for studying degeneration of the curve, that is, critical phenomena of QCD.

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A. The Weierstrass standard form and the period integrals

We write an elliptic curve in the Weierstrass standard form

$$y^2 = 4(t-e_1)(t-e_2)(t-e_3) \tag{A.1}$$

with

$$e_1 + e_2 + e_3 = 0\,.$$

Then the Jacobi θ-functions are given in terms of the positions of the branch points e_1, e_2 and e_3:

$$\begin{aligned}
\theta_2(\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} = i\left(\frac{2\omega_1}{\pi}\right)^{\frac{1}{2}}(e_2-e_3)^{\frac{1}{4}}\,,\\
\theta_3(\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} = \left(\frac{2\omega_1}{\pi}\right)^{\frac{1}{2}}(e_1-e_3)^{\frac{1}{4}}\,,\\
\theta_4(\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} = \left(\frac{2\omega_1}{\pi}\right)^{\frac{1}{2}}(e_1-e_2)^{\frac{1}{4}}\,,
\end{aligned} \tag{A.2}$$

with

$$\begin{aligned}
q &= e^{2\pi i\tau}\,,\\
\omega_1 &= \int_{-\infty}^{e_1} \frac{dt}{\sqrt{4(t-e_1)(t-e_2)(t-e_3)}}\,.
\end{aligned}$$

(A.1) may be also written in the form

$$y^2 = 4t^3 - g_2 t - g_3\,. \tag{A.3}$$

It is known that the coefficients are given by the Eisenstein series

$$
\begin{aligned}
g_2 &= 2(e_1^2+e_2^2+e_3^2) \\
&= \frac{4}{3}\left(\frac{\pi}{g_0}\right)^4 \left[1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\right] \equiv \frac{4}{3}\left(\frac{\pi}{g_0}\right)^4 E_4(\tau)\,, \qquad (A.4)\\
g_3 &= 4e_1e_2e_3 \\
&= \frac{8}{27}\left(\frac{\pi}{g_0}\right)^6 \left[1-504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}\right] \equiv \frac{8}{27}\left(\frac{\pi}{g_0}\right)^6 E_6(\tau)\,, \qquad (A.5)
\end{aligned}
$$

with $g_0 = \oint_A \frac{dt}{y} = -2\omega_1$. By using them we have the following integration formulae

$$
\begin{aligned}
&\frac{1}{g_0}\oint_A t\frac{dt}{y} = g_1\,, \qquad \frac{1}{g_0}\oint_B t\frac{dt}{y} = -\tau g_1 + \frac{2\pi i}{g_0^2}\,, && (A.6)\\
&\frac{1}{g_0}\oint_A t^2\frac{dt}{y} = \frac{g_2}{12}\,, \qquad \frac{1}{g_0}\oint_B t^2\frac{dt}{y} = \tau\frac{g_2}{12}\,, && (A.7)\\
&\frac{1}{g_0}\oint_A t^3\frac{dt}{y} = \frac{3}{20}g_2g_1 + \frac{g_3}{10}, \qquad \frac{1}{g_0}\oint_B t^3\frac{dt}{y} = \tau\left[\frac{3}{20}g_2g_1 + \frac{g_3}{10}\right], && (A.8)
\end{aligned}
$$

Here g_1 is also given by the Eisenstein series:

$$
g_1 = \frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 \left[1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right] \equiv -\frac{1}{3}\left(\frac{\pi}{g_0}\right)^2 E_2(\tau)\,. \qquad (A.9)
$$

(A.7) and (A.8) can be easily shown from

$$
\oint d(y) = 0\,, \qquad \oint d(ty) = 0\,.
$$

B. Calculations of $d\Omega_M$ and τ_{MN}

We will obtain a formula for the 2-point function τ_{MN}, directly calculating the residue of (2.11). First of all we expand $d\Omega_N$ given by (2.6) at $x = 1/\xi^2 = \infty$. Let the expansion of dx/y to be

$$
\frac{dx}{y} = -\sum_{M=1}^{\infty} R_M \xi^{2M-2} d\xi\,, \qquad (B.1)
$$

in which

$$
\begin{aligned}
R_1 &= 1\,, \qquad R_2 = \frac{c}{2}\,, \qquad R_3 = \frac{5c^2}{24} + \frac{1}{8}g_2\,,\\
R_4 &= \frac{35c^3}{432} + \frac{7c}{48}g_2 + \frac{1}{8}g_3\,,\\
R_5 &= \frac{35c^4}{1152} + \frac{7c^2}{64}g_2 + \frac{3c}{16}g_3 + \frac{3}{128}(g_2)^2\,,\\
R_6 &= \frac{77c^5}{6912} + \frac{77c^3}{1152}g_2 + \frac{11c^2}{64}g_3 + \frac{11c}{256}(g_2)^2 + \frac{3}{64}g_3g_2\,, \qquad \text{etc.,} \qquad (B.2)
\end{aligned}
$$

with c defined by (3.2). Then the requirement (2.8) gives the relation

$$ {}^t(\gamma_{N-1}\gamma_{N-2}\cdots\gamma_1) = -\mathcal{K}^{-1\,t}(R_2\cdots R_N)\,, \qquad N\geq 2 \tag{B.3}$$

with the matrix $\mathcal{K}$

$$\mathcal{K} = \begin{pmatrix} R_1 & & & \text{\Large 0} \\ R_2 & R_1 & & \\ \vdots & \vdots & \ddots & \\ R_{N-1} & R_{N-2} & \cdot\quad\cdot & R_1 \end{pmatrix}.$$

By (2.6) and (B.3), (2.11) is calculated as

$$\begin{aligned} \frac{2M-1}{2\pi i}\tau_{NM} &= -\mathrm{res}_{\xi=0}[\xi^{-2M+1}d\Omega_N] \\ &= R_{M+N} + \gamma_0 R_M - (R_{M+N-1}\cdots R_{M+1})\mathcal{K}^{-1\,t}(R_2\cdots R_N)\,, \end{aligned} \tag{B.4}$$

for $N\geq 2$ and $M\geq 1$. Here γ_0 is the constant in $d\Omega_N$ which is determined by (2.7). For $N=1$ and $M\geq 1$ we have

$$\begin{aligned} \frac{2M-1}{2\pi i}\tau_{1M} &= -\mathrm{res}_{\xi=0}[\xi^{-2M+1}d\Omega_1] \\ &= R_{M+1} - \left(g_1 + \frac{c}{3}\right) R_M\,, \end{aligned} \tag{B.5}$$

in which g_1 was given by (A.9). From $\tau_{1N}=\tau_{N1}$ we find alternatively γ_0 in (B.4) to be

$$\begin{aligned} \gamma_0 = &-\frac{2N-2}{2N-1}R_{N+1} - \frac{1}{2N-1}\left(g_1+\frac{c}{3}\right)R_N + \\ &+ (R_N\cdots R_2)\mathcal{K}^{-1\,t}(R_2\cdots R_N)\,, \qquad N=2,3,\ldots\,. \end{aligned} \tag{B.6}$$

For τ_{0N} with $M\geq 1$ we have

$$\frac{2M-1}{2\pi i}\tau_{0M} \equiv \frac{2M-1}{2\pi i}\tau_{M0} = \frac{R_M}{g_0}\,, \tag{B.7}$$

from (2.9) and (2.10). Evaluating the formulae (B.4), (B.5) and (B.7) for simple cases yields $\tau_{AB}(=\tau_{BA})$ with $A,B=0,1,2,\ldots$, as

$$\begin{aligned} \tau_{01} &= \frac{2\pi i}{g_0}\,, \qquad \tau_{02} = \frac{2\pi i}{g_0}\frac{c}{6}\,, \qquad \tau_{03} = \frac{2\pi i}{g_0}\left[\frac{c^2}{24}+\frac{g_2}{40}\right], \\ \tau_{11} &= 2\pi i\left[\frac{c}{6}-g_1\right], \\ \tau_{12} &= 2\pi i\left[\frac{c^2}{72}-\frac{c}{6}g_1+\frac{g_2}{24}\right], \\ \tau_{13} &= 2\pi i\left[\frac{13c^3}{432}+\frac{c^2}{24}g_1+\frac{3c}{80}g_2+\frac{g_3}{40}+\frac{g_1g_2}{40}\right], \\ \tau_{22} &= \pi i\left[\frac{c^3}{324}-\frac{c^2}{18}g_1+\frac{c}{36}g_2+\frac{1}{12}g_3\right], \\ \tau_{23} &= \pi i\left[\frac{c^4}{1728}-\frac{c^3}{72}g_1+\frac{c^2}{96}g_2+\frac{c}{20}(g_3-\frac{g_1g_2}{6})+\frac{(g_2)^2}{192}\right], \end{aligned}$$

$$\tau_{33} = \pi i\Bigg[\frac{c^5}{8640} - \frac{c^4}{288}g_1 + \frac{c^3}{288}g_2 + \frac{c^2}{240}(-g_2g_1 + 6g_3) + \\ + \frac{c}{192}(g_2)^2 + \frac{3}{400}g_3g_2 - \frac{1}{800}(g_2)^2g_1\Bigg]. \tag{B.8}$$

C. Consistency check

We shall alternatively check that (7.12) and (7.32) follow from from (7.13), although they are proved by (7.11) together with (7.10) and the calculation (7.16) respectively. It gives an independent consistency check of our formalism for the Whitham hierarchy.

C.1 Check of (7.12)

To this end we have to explicitly evaluate $s_M(\tau)$, given by (3.5). Let us put it in the form

$$s_M(\tau) = g_0\frac{\Gamma_M}{\Gamma_0}, \qquad M = 1, 2, \ldots, \tag{C.1}$$

with

$$\Gamma_M = u'Q_M(u)(v-w) + v'Q_M(v)(w-u) + w'Q(w)_M(u-v). \tag{C.2}$$

Writing the curve (2.1) in the Weierstrass standard form (A.1) we have

$$u(\tau) = e_1 + \frac{c}{3}, \qquad v(\tau) = e_2 + \frac{c}{3}, \qquad w(\tau) = e_3 + \frac{c}{3}. \tag{C.3}$$

Then (C.2) reads

$$\Gamma_M = \left(e_1' + \frac{c'}{3}\right)Q_M\left(e_1 + \frac{c}{3}\right)(e_2 - e_3) + \text{cyclic}. \tag{C.4}$$

First of all we replace $e_i'(= \partial e_i/\partial\tau)$ in (C.4) by

$$\frac{1}{2\pi i}e_i' = -\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2\left(e_i^2 + g_1e_i - \frac{g_2}{6}\right) - \frac{1}{\pi i}\frac{g_0'}{g_0}e_i, \tag{C.5}$$

which is a variant of the formula due to Itoyama and Morosov [7]

$$2\frac{\partial}{\partial\log q}\hat e_i = \frac{1}{6}\hat g_2(\tau) - \hat g_1(\tau)\hat e_i - \hat e_i^2, \qquad q = e^{2\pi i\tau}, \tag{C.6}$$

with

$$\begin{aligned}
\hat e_i &= \left(\frac{g_0}{\pi}\right)^2 e_i, \\
\hat g_1(\tau) &= -\frac{1}{3}E_2(\tau) = \left(\frac{g_0}{\pi}\right)^2 g_1, \\
\hat g_2(\tau) &= \frac{2}{3}[\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_0(\tau)^8] \\
&= \frac{4}{3}E_4(\tau) = \left(\frac{g_0}{\pi}\right)^4 g_2.
\end{aligned}$$

Here g_1 and g_2 are given by (A.9) and (A.4). Then we find

$$\begin{aligned}\frac{1}{2\pi i}\Gamma_M &= \left[-\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2\left(e_1^2+g_1e_1-\frac{g_2}{6}\right)-\frac{1}{\pi i}\frac{g_0'}{g_0}e_1+\frac{1}{2\pi i}\frac{c'}{3}\right]\times\\ &\quad\times Q_M\left(e_1+\frac{c}{3}\right)(e_2-e_3)+\text{cyclic}\,. \end{aligned}\tag{C.7}$$

Next we replace $Q_M(e_i+\frac{c}{3}), i=1,2,3$, in the r.h.s. by those calculated in (7.20), (7.24) and (7.28). The resultant Γ_M is simplified by the formulae

$$\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2 e_1^2(e_2-e_3)+\text{cyclic} = -\frac{\Gamma_0}{2\pi i}\,,\tag{C.8}$$

$$\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2 e_1^3(e_2-e_3)+\text{cyclic} = 0\,,\tag{C.9}$$

$$\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2 e_1^4(e_2-e_3)+\text{cyclic} = -\frac{\Gamma_0}{2\pi i}\frac{g_2}{4}\,,\tag{C.10}$$

$$\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2 e_1^5(e_2-e_3)+\text{cyclic} = -\frac{\Gamma_0}{2\pi i}\frac{g_3}{4}\,,\tag{C.11}$$

$$\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2 e_1^6(e_2-e_3)+\text{cyclic} = -\frac{\Gamma_0}{2\pi i}\left(\frac{1}{16}(g_2)^2+\frac{1}{2}g_1g_3\right),\tag{C.12}$$

$$\cdots$$

Finally we eliminate $c'(=\partial c/\partial\tau)$ from Γ_M for the cases with $M_0=1, M_0=2$ and $M_0=3$ by means of the constraints (7.17)–(7.19) respectively. To this end we need to differentiate the constraints by τ using the formulae

$$\frac{\partial}{\partial\tau}E_2(\tau) = \frac{i\pi}{6}(E_2(\tau)^2-E_4(\tau))\,,\tag{C.13}$$

$$\frac{\partial}{\partial\tau}E_4(\tau) = \frac{2\pi i}{3}(E_2(\tau)E_4-E_6(\tau))\,,\tag{C.14}$$

$$\frac{\partial}{\partial\tau}E_6(\tau) = -i\pi(E_4(\tau)^2-E_2(\tau)E_6(\tau))\,,\tag{C.15}$$

$$\cdots$$

We have checked that the characteristic speeds (C.1) with Γ_M thus evaluated indeed satisfy (7.12) for the respective case of $M_0=1, M_0=2$ and $M_0=3$.

The formulae (C.8)–(C.12), (C.13)–(C.15) can be shown by means of (C.5). First of all we show (C.8)–(C.12). Integrating (3.9) along a A-cycle in the Weierstrass standard form we obtain

$$\oint_A\frac{1}{t-e_1}\frac{dt}{y}=\frac{g_0}{\Gamma_0}(e_2'-e_3')+2\frac{e_2-e_3}{\Gamma_0}g_0'\,.$$

By means of (C.5) it becomes

$$\frac{1}{2\pi i}\oint_A\frac{1}{t-e_1}\frac{dt}{y}=\frac{1}{2}\left(\frac{g_0}{\pi}\right)^2\frac{g_0}{\Gamma_0}(e_2-e_3)(e_1-g_1)\,.\tag{C.16}$$

The similar formulae are found by cyclic rotation of e_1, e_2 and e_3. Now we calculate $d(\frac{t^n}{y})$ for $n=1,2,3,4,5$ as

$$d\left(\frac{t}{y}\right) = \left[-\frac{1}{2}-\frac{1}{2}\sum_i\frac{e_i}{t-e_i}\right]\frac{dt}{y}\,,$$

$$d\left(\frac{t^2}{y}\right) = \left[\frac{t}{2} - \frac{1}{2}\sum_i \frac{e_i^2}{t-e_i}\right]\frac{dt}{y}.$$
$$\cdots$$

Integrate them along a A-cycle and use (C.16). With (A.4)–(A.9) we then obtain the formulae (C.8)–(C.12).

Next we go to the proof of (C.13)–(C.15). Either of them is proved similarly. We will sketch a proof of (C.14). Differentiate the first equation of (A.7) by τ:

$$\left(\frac{g_2}{12}\right)' = -\frac{g_0'}{g_0}\frac{g_2}{12} + \frac{1}{2g_0}\sum_i e_i' \oint_A \left[t + e_i + \frac{e_i^2}{t-e_i}\right]\frac{dt}{y}. \tag{C.17}$$

We calculate the second piece of the r.h.s. by the following three steps: At first perform the integration by (A.6)–(A.8) and (C.16). Next replace e_i' by the formula (C.5). Finally sum over i by (C.8)–(C.11). Then we find

$$\frac{1}{2g_0}\sum_i e_i' \oint_A \left[t + e_i + \frac{e_i^2}{t-e_i}\right]\frac{dt}{y} = -\frac{\pi i}{2}\left(\frac{g_0}{\pi}\right)^2\left[\frac{g_3}{2} + \frac{g_1 g_2}{3}\right] - \frac{g_0'}{g_0}\frac{g_2}{4}. \tag{C.18}$$

On the other hand the l.h.s. of (C.17) is calculated as

$$\left(\frac{g_2}{12}\right)' = -\frac{g_0'}{g_0}\frac{g_2}{3} + \frac{1}{9}\left(\frac{\pi}{g_0}\right)^4 E_4(\tau)', \tag{C.19}$$

by (A.4). Putting (C.18) and (C.19) into (C.17) and using the Eisenstein series, we obtain (C.14).

C.2 Check of (7.32)

We check (7.32) for case ii), i.e.,

$$\Lambda_2^2 \frac{\partial}{\partial\tau}\tilde{\tau}_{22} = \left[\frac{\pi}{72}(E_2(\tau)^2 - E_4(\tau))\right]^2. \tag{C.20}$$

For other cases it can be done similarly. In the first place we compute the residue in $\tilde{\tau}_{22}$ given by (7.26):

$$\begin{aligned} 2\pi i\ \mathrm{res}_{\xi=0}[\xi^{-4} d\xi \Delta_2(x,\tau)] = \pi i\Big[&-\frac{1}{\Gamma_0}\{Q_2(u)u'(v'w - w'v) + \text{cyclic}\} + \\ &+ \frac{c}{2\Gamma_0}\{Q_2(u)u'(v'-w') + \text{cyclic}\}\Big], \end{aligned} \tag{C.21}$$

using (3.6) and (B.1). By (C.3) both cyclic sums can be written as

$$\begin{aligned} Q_2(u)u'(v'w - w'v) + \text{cyclic} &= Q_2\left(e_1 + \frac{c}{3}\right)\left(e_1' + \frac{c'}{3}\right)\times \\ &\quad\times\left[(e_2'e_3 - e_3'e_2) + \frac{c}{3}(e_2' - e_3') - \frac{c'}{3}(e_2 - e_3)\right], \\ Q_2(u)u'(v'-w') + \text{cyclic} &= Q_2\left(e_1 + \frac{c}{3}\right)\left(e_1' + \frac{c'}{3}\right)(e_2' - e_3'). \end{aligned}$$

With recourse to (C.5) and (C.8)–(C.12) they are evaluated in the same way as Γ_M, (C.4). After tedious calculations we find that

$$
\begin{aligned}
2\pi i\ \mathrm{res}_{\xi=0}[\xi^{-4}d\xi\Delta_2(x,\tau)] = {} & (\pi i)^2\left(\frac{g_0}{\pi}\right)^2\times \\
& \times\left[\frac{g_2^2}{72}+\frac{g_1g_3}{4}++c\left(\frac{g_1g_2}{18}+\frac{g_3}{12}\right)+c^2\left(\frac{g_2}{432}-\frac{(g_1)^2}{36}\right)\right]+ \\
& +2\pi i\frac{g_0'}{g_0}\left[\frac{g_3}{4}+\frac{c}{18}g_2-\frac{c^2}{18}g_1\right]+\pi ic'\left[-\frac{g_2}{36}+\frac{c}{9}g_1-\frac{c^2}{108}\right]- \\
& -\left(\frac{\pi}{g_0}\right)^2\left[\frac{1}{3}\left(c'-c\frac{g_0'}{g_0}\right)\right]^2. \qquad \text{(C.22)}
\end{aligned}
$$

On the other hand we calculate $\frac{\partial\tau_{22}}{\partial\tau}$ by differentiating directly the formula given by (B.8). Then add the result to (C.22). It exactly cancels the first three terms of (C.22). We use the constraint (7.23) and (C.13) in the last term to obtain (C.20).

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